NOTE ON THE USE OF GROUP THEORY IN ELEMENTARY TRIGONOMETRY

By G. A. Miller

1. The applications of group theory in the development of some of the fundamental parts of trigonometry have received considerable attention ;* while some of these admit of interesting amplifications yet most of them relate to spherical trigonometry and are too complex to be immediately useful to the average teacher of elementary trigonometry. The present note relates to developments which are generally treated near the beginning of a course in trigonometry. While the mode of treatment which is outlined may not be suitable for the beginner, it will doubtless enable the teacher to see many points in a new light and to observe relations which had escaped notice.

The only group properties which are employed are those which belong to the cyclic group and the dihedral rotation group. While a knowledge of these groups will make the reading easier, yet it is not essential, in view of the elementary character of the considerations. In fact, these trigonometric developments furnish one of the easiest means of acquiring a clear notion of the groups in question. This paper has very close contact with the writer's paper entitled "A new chapter in trigonometry"† and includes the results developed in the first part of that paper.

* E. Study, *Abhandlungen der Sächsischen Gesellschaft der Wissenschaften*, vol. 20 (1893), p. 83; F. Meyer, *Crelle*, vol. 115 (1895), p. 209; G. Chisholm, *Dissertation*, Göttingen, 1895; O. Pund, *Hamburger math. Gesellschaft*, vol. 3 (1897), p. 290; G. A. Miller, *Quarterly Journal of Mathematics*, vol. 37 (1906), p. 226.

† *Quarterly Journal of Mathematics*, loc. cit.

We assume as known that

$$\sin(90^\circ - x) = \cos x, \qquad \cos(90^\circ - x) = \sin x,$$

$$\sin(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x - \sin x), \quad \cos(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x + \sin x).$$

It is proposed to derive from these formulas, with the help of some elementary group theory, expressions for the functions of fourteen other angles in terms of those of x.

The operation of deriving the angle $45^\circ - x$ from x is of order 2, since $45^\circ - (45^\circ - x) = x$. Representing this operation by t and that of taking the complement of any angle by c, we have $t^2 = c^2 = 1$. The group $\{t, c\}$ generated by t and c is therefore of the dihedral rotation type and its order is twice the order of tc.* As tc signifies that an angle is subtracted from 45° and the complement of the resulting angle is obtained, it follows that $tc\,x = x + 45^\circ$, and therefore tc is of order 8. Hence $\{t, c\}$ is of order 16.

If $tc = t_1$, the operations of this group may be written as follows:

1	t_1^2	t_1^4	t_1^6	t_1	t_1^3	t_1^5	t_1^7
c	t_1^2c	t_1^4c	t_1^6c	t_1c	t_1^3c	t_1^5c	t_1^7c

All the operations of the second row are of order 2 while those of the first constitute the cyclic group of order 8. The operations of the first four columns constitute the group of order 8 known as the octic group or the group of the square. This subgroup is generated by c and $t_1^6c = s$, where s represents the operation of taking the supplement of an angle. If we apply these 16 operations to x the resulting angles in order (since angles are taken modulo 2π) are as follows:

x	$x + 90^\circ$	$x + 180^\circ$	$x + 270^\circ$	$x + 45^\circ$	$x + 135^\circ$	$x + 225^\circ$	$x + 315^\circ$
$90^\circ - x$	$-x$	$270^\circ - x$	$180^\circ - x$	$45^\circ - x$	$315^\circ - x$	$225^\circ - x$	$135^\circ - x$

If these 16 operations were applied to any other one of these angles the same 16 angles would result (in different arrangement), since the sixteen operations constitute a group.

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* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

2. The trigonometric functions of sixteen angles. What precedes has been preliminary to expressing the trigonometric functions of each of these sixteen angles in terms of those of x assuming only that the equations giving the functions of $45° - x$ and of $90° - x$ in terms of those of x are known. It will be sufficient to do this for the sine and the cosine since the other functions are readily derived from these two. We begin with $x + 90°$ and for clearness give all the details in this instance.

$$\begin{aligned}\sin(x + 90°) &= \sin t_1^2 x = \sin tc\, tcx = \cos tctx = \tfrac{1}{2}\sqrt{2}(\cos tcx + \sin tcx)\\ &= \tfrac{1}{2}\sqrt{2}(\sin tx + \cos tx) = \tfrac{1}{2}(\cos x - \sin x + \cos x + \sin x) = \cos x,\\ \cos(x + 90°) &= \cos t_1^2 x = \sin tctx = \tfrac{1}{2}\sqrt{2}(\cos tcx = \sin tcx) = \tfrac{1}{2}\sqrt{2}(\sin tx - \cos tx)\\ &= \tfrac{1}{2}(\cos x - \sin x - \cos x - \sin x) = - \sin x.\end{aligned}$$

The functions of the angles corresponding to t_1^4 and t_1^6 may be directly derived from the preceding results as follows:

$$\begin{aligned}\sin(x + 180°) &= \sin t_1^4 x = \sin t_1^2 t_1^2 x = \cos t_1^2 x = - \sin x,\\ \cos(x + 180°) &= \cos t_1^4 x = \cos t_1^2 t_1^2 x = - \sin t_1^2 x = - \cos x;\\ \sin(x + 270°) &= \sin t_1^6 x = \sin t_1^2 t_1^4 x = - \sin t_1^2 x = - \cos x,\\ \cos(x + 270°) &= \cos t_1^6 x = \cos t_1^4 t_1^2 x = - \sin t_1^4 x = \sin x.\end{aligned}$$

The functions of the four angles corresponding to c, $t_1^2 c$, $t_1^4 c$, $t_1^6 c$ can be directly written down by means of our knowledge of the effect of the fundamental operation c and the results which have just been obtained. The work may be done as follows:

$$\begin{aligned}\sin(90° - x) &= \sin cx = \cos x,\\ \cos(90° - x) &= \cos cx = \sin x;\\ \sin(- x) &= \sin t_1^2 cx = \cos t_1^2 x = - \sin x,\\ \cos(- x) &= \cos t_1^2 cx = \sin t_1^2 x = \cos x;\\ \sin(270° - x) &= \sin t_1^4 cx = \cos t_1^4 x = - \cos x,\\ \cos(270° - x) &= \cos t_1^4 cx = \sin t_1^4 x = - \sin x;\\ \sin(180° - x) &= \sin t_1^6 cx = \cos t_1^6 x = \sin x,\\ \cos(180° - x) &= \cos t_1^6 cx = \sin t_1^6 x = - \cos x.\end{aligned}$$

The functions of the three angles corresponding to t_1^3, t_1^5, t_1^7 may be expressed as follows :

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\begin{aligned}
\sin(x+135^\circ) &= \sin t_1^3x = \sin t_1t_1^2x = \cos t_1x = \sin tx = \tfrac{1}{2}\sqrt{2}(\cos x - \sin x),\\
\cos(x+135^\circ) &= \cos t_1^3x = \cos t_1t_1^2x = -\sin t_1x = -\cos tx = -\tfrac{1}{2}\sqrt{2}(\cos x + \sin x);\\
\sin(x+225^\circ) &= \sin t_1^5x = \sin t_1t_1^4x = -\sin t_1x = -\cos tx = -\tfrac{1}{2}\sqrt{2}(\cos x + \sin x),\\
\cos(x+225^\circ) &= \cos t_1^5x = \cos t_1t_1^4x = -\cos t_1x = -\sin tx = \tfrac{1}{2}\sqrt{2}(\sin x - \cos x);\\
\sin(x+315^\circ) &= \sin t_1^7x = \sin t_1t_1^6x = -\cos t_1x = -\sin tx = \tfrac{1}{2}\sqrt{2}(\sin x - \cos x),\\
\cos(x+315^\circ) &= \cos t_1^7x = \cos t_1t_1^6x = \sin t_1x = \cos tx = \tfrac{1}{2}\sqrt{2}(\cos x + \sin x).
\end{aligned}
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As the functions of the angles which correspond to the remaining operations of our group may be obtained by first passing to the co-function and then proceeding as above it seems unnecessary to give them here. The thing which we desire to emphasize is the *method* of deriving these functions and that a knowledge of this group of order 16 gives us a clear insight into the number of possible similar ways of attaining the same results. Since this group is generated by 47 other pairs of operations we could have chosen the two fundamental angles in 48 distinct ways * and the derivation of the trigonometric functions of the remaining fourteen angles could have proceeded along the same lines. This clear comprehension of the various roads which lead to the same goal combined with the freedom to choose from such a variety of almost equally good roads constitutes one of the greatest sources of pleasure to the appreciative traveler in a mathematical field.

If it had been desired to find only the functions of the angles corresponding to the octic group we could have used the angles which correspond to two of its generators as fundamental instead of those corresponding to c and t. This was done in the article published in the *Quarterly Journal* to which we have already referred. The method pursued here presents the matter under a more comprehensive aspect and exhibits more extensive relations. There is however no limit to the generalizations along this line. For instance, if the operation t were replaced by an operation r, where rx indicates the result of subtracting the angle x from 30° : $rx = 30^\circ - x$, we should have to deal with a dihedral rotation group of order 24 which would again contain the octic group

* The number of ways of choosing two generators of the dihedral rotation group of order g is $\frac{3}{4}g\,\phi\,(\frac{1}{2}g)$, $\phi(n)$ being the totient, or indicator, of n.

as a subgroup. The considerations with respect to this group are practically the same as those given above.

3. The trigonometric functions of forty-eight angles. Since $rtx = x + 15°$, its order is 24 and $\{r, t\}$ is the dihedral rotation group of order 48. The angles which correspond to this group are x increased by the twenty-four multiples of 15° together with their negatives. The functions of these 48 angles can therefore be directly expressed in terms of those of x provided we know those of the angles corresponding to generators of $\{r, t\}$. As we have assumed a knowledge of those which correspond to t it is only necessary to add those which correspond to r; i. e.

$$\sin(30° - x) = \tfrac{1}{2}(\cos x - \sqrt{3}\sin x), \qquad \cos(30° - x) = \tfrac{1}{2}(\sqrt{3}\cos x + \sin x).$$

Hence,

$$\begin{aligned}
\sin(x + 15°) &= \sin rtx = \tfrac{1}{2}\sqrt{2}\,(\cos rx - \sin rx) \\
&= \tfrac{1}{4}\sqrt{2}(\sqrt{3}\cos x + \sin x - \cos x + \sqrt{3}\sin x) \\
&= \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin x + (\sqrt{3}-1)\cos x\}, \\
\cos(x + 15°) &= \cos rtx = \tfrac{1}{2}\sqrt{2}\,(\cos rx + \sin rx) \\
&= \tfrac{1}{4}\sqrt{2}(\sqrt{3}\cos x + \sin x + \cos x - \sqrt{3}\sin x) \\
&= \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos x - (\sqrt{3}-1)\sin x\}.
\end{aligned}$$

The functions of the remaining 47 angles may be readily found from these results if we observe that $\{r, t\} \equiv \{rt, c\}$, and hence $rt = t_2$ and c may be used as the fundamental operations. To obtain the functions of $75° - x = rtcx$ it is only necessary to observe that $\sin rtcx = \cos rtx$ and $\cos rtcx = \sin rtx$, from which it follows that

$$\sin(75° - x) = \sin t_2cx = \cos t_2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos x - (\sqrt{3}-1)\sin x\},$$

$$\cos(75° - x) = \cos t_2cx = \sin t_2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin x + (\sqrt{3}-1)\cos x\}.$$

Further,

$$\sin(x + 30°) = \sin t_2^2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin t_2x + (\sqrt{3}-1)\cos t_2x\}$$
$$= \tfrac{1}{8}\{(4+2\sqrt{3})\sin x + 2\cos x + 2\cos x - (4-2\sqrt{3})\sin x\} = \tfrac{1}{2}(\cos x + \sqrt{3}\sin x),$$
$$\cos(x + 30°) = \cos t_2^2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos t_2x - (\sqrt{3}-1)\sin t_2x\}$$
$$= \tfrac{1}{8}\{(4+2\sqrt{3})\cos x - 2\sin x - 2\sin x - (4-2\sqrt{3})\cos x\} = \tfrac{1}{2}(\sqrt{3}\cos x - \sin x);$$

$$
\begin{aligned}
\sin(x+60^\circ) &= \sin t_2^4 x = \tfrac{1}{2}(\cos t_2^2 x + \sqrt{3}\sin t_2^2 x) \\
&= \tfrac{1}{4}(\sqrt{3}\cos x - \sin x + \sqrt{3}\cos x + 3\sin x) = \tfrac{1}{2}\sqrt{3}\cos x + \sin x), \\
\cos(x+60^\circ) &= \cos t_2^4 x = \tfrac{1}{2}(\sqrt{3}\cos t_2^2 x - \sin t_2^2 x) \\
&= \tfrac{1}{4}(3\cos x - \sqrt{3}\sin x - \cos x - \sqrt{3}\sin x) = \tfrac{1}{2}(\cos x - \sqrt{3}\sin x).
\end{aligned}
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The preceding examples appear ample to exhibit the method of work and the numerous relations which it sets forth. The two generating operations of $\{r, t\}$ could be selected in $36 \cdot 8 = 288$ different ways so that there is an almost endless variety of ways from which to choose in deriving the functions of these 48 angles along this line. The method aids in deducing a great deal from a few known facts but is of no value without such knowledge. It has a theoretic and clarifying influence but does not seem to fit in a brief practical course in trigonometry. If the trigonometric functions of a series of angles is known, those of any multiple (positive or negative) of the highest common factor of all the possible differences between these angles may be found in this way.* In fact, we are dealing simply with the group of subtraction from several fixed numbers with respect to modulus 2π, which is considered in the last article to which we have referred. The present article may serve as an illustration of the utility of such groups.

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* ANNALS OF MATHEMATICS, ser. 2, vol. 6 (1905), p. 41.